ON RADICALLY GRADED FINITE DIMENSIONAL QUASI-HOPF ALGEBRAS

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Abstract. In this paper we continue the structure theory of finite dimensional quasi-Hopf algebras started in [EG] and [G]. First, we completely describe the class of radically graded finite dimensional quasi-Hopf algebras over $\mathbb{C}$, whose radical has prime codimension. As a corollary we obtain that if $p > 2$ is a prime then any finite tensor category over $\mathbb{C}$ with exactly p simple objects which are all invertible must have Frobenius-Perron dimension p^N, $N = 1, 2, 3, 4, 5$ or 7. Second, we construct new examples of finite dimensional quasi-Hopf algebras which are not twist equivalent to a Hopf algebra. For instance, to every finite dimensional simple Lie algebra $\mathfrak{g}$ and an odd integer n, coprime to 3 if $\mathfrak{g} = G_2$, we attach a quasi-Hopf algebra of dimension $n^{\dim \mathfrak{g}}$.

1. Introduction

In [EO] it is proved that any finite tensor category over $\mathbb{C}$ with integer Frobenius-Perron dimensions of objects is equivalent to a representation category of a finite dimensional quasi-Hopf algebra (the Frobenius-Perron dimension of a representation coincides with its dimension as a vector space). Therefore the classification of finite tensor categories with integer Frobenius-Perron dimensions of objects is equivalent to the classification of complex finite dimensional quasi-Hopf algebras. The simplest finite tensor categories to try to understand are those which have only 1-dimensional simple objects which form a cyclic group of prime order under tensor product. Equivalently, one is led to the problem of classifying finite dimensional quasi-Hopf algebras with (Jacobson) radical of prime codimension.

Let p be a prime, and let $RG(p)$ denote the class of radically graded finite dimensional quasi-Hopf algebras over $\mathbb{C}$, whose radical has codimension p. It was shown in [EG] that any $H \in RG(2)$ is equivalent to a Nichols Hopf algebra H_{2^n}, $n \geq 1$ [N], or to a lifting of one of the four special quasi-Hopf algebras $H(2)$, $H_+(8)$, $H_-(8)$, $H(32)$ of dimensions 2, 8, 8, and 32 (the algebra $H(2)$ is the group algebra of $\mathbb{Z}_2$ with a nontrivial associator).

Later, it was shown in [G] that if $H \in RG(p)$, $p > 2$, has a nontrivial associator and if the rank of $H[1]$ over $H[0]$ is 1, then H is equivalent to one of the quasi-Hopf algebras $A(q)$ of dimension p^3, introduced in [G]. More precisely, the result of [G] is formulated under the assumption that H is basic (i.e., $H/\mathrm{Rad}(H) = \mathbb{C}[\mathbb{Z}_p]$ with some associator), but by [ENO], Corollary 8.31, this is automatic.

The purpose of this paper is to continue the structure theory of finite dimensional quasi-Hopf algebras started in [EG] and [G]. More specifically, we completely describe the class $RG(p)$, and construct new examples of finite dimensional quasi-Hopf algebras which are not twist equivalent to a Hopf algebra.

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The structure of the paper is as follows. In Section 2 we recall the definition of the quasi-Hopf algebras $A(q)$ and $H_{\pm}(p)$.

In Section 3 we show that if $H \in RG(p)$ has a nontrivial associator, then the rank of $H[1]$ over $H[0]$ is ≤ 1. This yields the following classification of $H \in RG(p)$, $p > 2$, up to twist equivalence.

(a) Duals of pointed Hopf algebras with p grouplike elements, classified in [AS], Theorem 1.3.

(b) Group algebra of $\mathbb{Z}_p$ with associator defined by a 3-cocycle.

(c) The algebras $A(q)$.

This result implies, in particular, that if $p > 2$ is a prime then any finite tensor category over $\mathbb{C}$ with exactly p simple objects which are all invertible must have Frobenius-Perron dimension p^N, $N = 1, 2, 3, 4, 5$ or 7.

In Section 4 we construct new examples of finite dimensional quasi-Hopf algebras H, which are not twist equivalent to a Hopf algebra. They are radically graded, and $H/\mathrm{Rad}(H) = \mathbb{C}[\mathbb{Z}_n^m]$, with a nontrivial associator. For instance, to every finite dimensional simple Lie algebra $\mathfrak{g}$ and an odd integer n, coprime to 3 if $\mathfrak{g} = G_2$, we attach a quasi-Hopf algebra of dimension $n^{\dim \mathfrak{g}}$.

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2. Preliminaries

All constructions in this paper are done over the field of complex numbers $\mathbb{C}$.

We refer the reader to [D] for the definition of a quasi-Hopf algebra and a twist of a quasi-Hopf algebra.

2.1. We recall the theory of the radical filtration for finite dimensional quasi-Hopf algebras, discussed in [EG]. It is completely parallel to the classical theory of such filtration in finite dimensional Hopf algebras.

Let H be a finite dimensional quasi-Hopf algebra, and I be the radical of H. Assume that I is a quasi-Hopf ideal, i.e., $\Delta(I) \subseteq H \otimes I + I \otimes H$. In categorical terms, this means that the category of representations $\mathrm{Rep}(H)$ has Chevalley property, i.e., the tensor product of irreducible H-modules is completely reducible. This is satisfied, for example, if H is basic, i.e., every irreducible H-module is 1-dimensional.

In this situation, the filtration of H by powers of I is a quasi-Hopf algebra filtration. Thus the associated graded algebra $\mathrm{gr}(H)$ of H under this filtration has a natural structure of a quasi-Hopf algebra.

Let now $\overline{H}$ be a finite dimensional quasi-Hopf algebra with a $\mathbb{Z}_+$-grading, i.e., $\overline{H} = \oplus_{m \ge 0} \overline{H}[m]$, with all structure maps being of degree zero. In this case, $\overline{H}[0]$ is a quasi-Hopf algebra, $\overline{H}[i]$ is a free module over $\overline{H}[0]$ for all i (by Schauenburg's theorem [S]), and the radical $\overline{I}$ of $\overline{H}$ is a quasi-Hopf ideal.

One says that $\overline{H}$ is radically graded if $\overline{I}^k = \oplus_{m \ge k} \overline{H}[m]$, for $k \ge 1$. In this case, $\overline{H}[0]$ is semisimple, and $\overline{H}$ is generated by $\overline{H}[0]$ and $\overline{H}[1]$.

An example of a radically graded quasi-Hopf algebra is the algebra $\mathrm{gr}(H)$ defined above. Moreover, H is radically graded if and only if $\mathrm{gr}(H) = H$.

Finally, we observe that if H is radically graded and basic, then $H[0] = \mathrm{Fun}(G)$ for a finite group G, and the associator (being of degree zero) corresponds to a class in $H^3(G, \mathbb{C}^*)$.

2.2. The following are the simplest examples of quasi-Hopf algebras not twist equivalent to a Hopf algebra.

Let $p > 2$ be a prime, and $\varepsilon = e^{2\pi \mathrm{i}/p}$. If $z \in \mathbb{Z}$, we denote by z' the projection of z to $\mathbb{Z}_p$.

Let s be an integer such that $1 \le s \le p-1$. Let $Q = \varepsilon^{-s}$. The $p-$dimensional quasi-Hopf algebra $H(p,s)$, is generated by a grouplike element a such that $a^p = 1$, with non-trivial associator

$$\Phi_s := \sum_{i,j,k=0}^{p-1} Q^{\frac{-i(j+k-(j+k)')}{p}} \mathbf{1}_i \otimes \mathbf{1}_j \otimes \mathbf{1}_k \tag{1}$$

where $\{\mathbf{1}_i | 0 \le i \le p-1\}$ is the set of primitive idempotents of $\mathbb{Z}_p$ (i.e, $\mathbf{1}_i a = Q^i \mathbf{1}_i$), distinguished elements $\alpha = a$, $\beta = 1$, and antipode $S(a) = a^{-1}$.

Let $s_0 \in \mathbb{Z}_p$ be a quadratic nonresidue. It can be shown (by considering automorphisms $a \mapsto a^m$) that for any s, $H(p,s)$ is isomorphic to $H_+(p) := H(p,1)$ if s is a quadratic residue, and to $H_-(p) := H(p, s_0)$ if s is a non-quadratic residue. On the other hand, $H_+(p)$ and $H_-(p)$ are not equivalent.

Thus it follows from [ENO], Corollary 8.31, that any $p-$dimensional semisimple quasi-Hopf algebra is twist equivalent either to $\mathbb{C}[\mathbb{Z}_p]$ or to $H_\pm(p)$.

2.3. The following are examples of p^3-dimensional basic quasi-Hopf algebras with radical of codimension p, which are not twist equivalent to a Hopf algebra.

Theorem 2.1. [G] *Let p be a prime number.*

(i) There exist p^3-dimensional quasi-Hopf algebras $A(q)$, parametrized by primitive roots of unity q of order p^2, which have the following structure. As algebras $A(q)$ are generated by a, x with the relations $ax = q^p xa$, $a^p = 1$, $x^{p^2} = 0$. The element a is grouplike, while the coproduct of x is given by the formula

$$\Delta(x) = x \otimes \sum_{y=0}^{p-1} q^y \mathbf{1}_y + 1 \otimes (1 - \mathbf{1}_0)x + a^{-1} \otimes \mathbf{1}_0 x,$$

where $\{\mathbf{1}_i | 0 \le i \le p-1\}$ is the set of primitive idempotents of $\mathbb{C}[a]$ defined by the condition $a\mathbf{1}_i = q^{pi}\mathbf{1}_i$, the associator is Φ_s (where s is defined by the equation $\varepsilon^{-s} = q^p$), the distinguished elements are $\alpha = a$, $\beta = 1$, and the antipode is $S(a) = a^{-1}$, $S(x) = -x \sum_{z=0}^{p-1} q^{p-z} \mathbf{1}_z$.

(ii) The quasi-Hopf algebras $A(q)$ are pairwise non-equivalent. Any finite dimensional radically graded basic quasi-Hopf algebra H with radical of codimension p and nontrivial associator, such that $H[1]$ is a free module of rank 1 over $H[0]$, is equivalent to $A(q)$ for some q.

3. QUASI-HOPF ALGEBRAS WITH RADICAL OF PRIME CODIMENSION

3.1. **The main result.** Let $p > 2$ be a prime number. Our main result in this section is the following theorem.

Theorem 3.1. *Let H be a radically graded basic quasi-Hopf algebra with radical of codimension p. If the associator of H is nontrivial, then the rank of $H[1]$ over $H[0]$ is ≤ 1.*

Theorem 3.1 is proved in the next subsection.

Theorem 3.1 and the results cited above imply the following classification result.

Theorem 3.2. *Let H be a radically graded finite dimensional quasi-Hopf algebra with radical of codimension p. Then H is one of the following quasi-Hopf algebras, up to twist equivalence:*

(a) Duals of pointed Hopf algebras with p grouplike elements, classified in [AS], *Theorem 1.3 (including the group algebra $\mathbb{C}[\mathbb{Z}_p]$).*

(b) The algebras $H_+(p)$ and $H_-(p)$.

(c) The algebras $A(q)$.

Proof. By Corollary 8.31 of [ENO], H is necessarily basic.

If the associator of H is trivial, then we may assume that H is a Hopf algebra. Thus H^* is a coradically graded pointed Hopf algebra with $G(H^*) = \mathbb{Z}_p$. Such algebras are classified in [AS], Theorem 1.3, so we are in case (a).

If the associator is nontrivial, then by Theorem 3.1, the rank of $H[1]$ over $H[0]$ is at most 1. If the rank is 0, we are in case (b). If the rank is 1, we are in case (c) by Theorem 2.1. □

We refer the reader to [EO], for the definition of a finite tensor category and the notion of its Frobenius- Perron dimension.

Corollary 3.3. *Let $p > 2$ be a prime. Let $\mathcal{C}$ be a finite tensor category, which has exactly p simple objects which are all invertible. Then the possible values of the Frobenius-Perron dimension of $\mathcal{C}$ are p^N, $N = 1, 2, 3$ (for all p), 4 (for $p = 3$ and $p = 3k + 1$), 5 (for $p = 3$ and $p = 4k + 1$) and 7 (for $p = 3$ and $p = 3k + 1$).*

Proof. It is clear that the Frobenius-Perron dimension of objects in $\mathcal{C}$ are integers. Hence by [EO], there exists a quasi-Hopf algebra A such that $\mathcal{C} = \mathrm{Rep}(A)$. This quasi-Hopf algebra is basic, so its radical is a quasi-Hopf ideal and hence A admits a radical filtration. Let $H := \mathrm{gr}(A)$ (with respect to this filtration). Then Theorem 3.2 applies to H, hence the result. □

3.2. **Proof of Theorem 3.1.** Let us assume that $H[1]$ has rank > 1 over $H[0]$. From this we will derive a contradiction. We may assume that H has the minimal possible dimension.

Let a be a generator of $\mathbb{Z}_p$. We have $H[0] = \mathbb{C}[\mathbb{Z}_p]$ with associator Φ_s for some s.

Let us decompose $H[1]$ into a direct sum of eigenspaces of a: $H[1] = \bigoplus_{r=0}^{p-1} H_r[1]$, where $H_r[1]$ is the space of $x \in H[1]$ such that $axa^{-1} = Q^r x$ (we recall that $Q := \varepsilon^{-s}$). Note that $\mathbf{1}_i x = x\mathbf{1}_{i-r}$ for $x \in H_r[1]$. Also, by Theorem 2.17 in [EO], $H_0[1] = 0$.

Let $\tilde{H}$ be the free algebra generated by $H[1]$ as a bimodule over $H[0]$; i.e., $\tilde{H}$ is the tensor algebra of $H[1]$ over $H[0]$. Then $\tilde{H}$ is (an infinite dimensional) quasi-Hopf algebra, and we have a surjective homomorphism $\varphi : \tilde{H} \to H$ (it is surjective since H is radically graded and hence generated by $H[0]$ and $H[1]$).

Let q be a number such that $q^p = Q$. Define an automorphism γ of $\tilde{H}$ by the formula $\gamma_{|H[0]} = 1$ and $\gamma_{|H_r[1]} = q^r$. (It is well defined since $\tilde{H}$ is free.)

Let L be the sum of all quasi-Hopf ideals in $\tilde{H}$ contained in $\bigoplus_{d\geq 2} \tilde{H}[d]$. Clearly, $Ker\varphi \subseteq L$, so H projects onto $\tilde{H}/L$. However, since H has the smallest dimension, it follows that $\tilde{H}/L = H$.

Now, $\gamma(L) = L$, so γ acts on H. Let us define a new quasi-Hopf algebra $\hat{H}$ generated by H and a grouplike element g with relations $g^p = a$, $gzg^{-1} = \gamma(z)$ for $z \in H$. Clearly, $\mathrm{Ad}(a) = \gamma^p$, and g generates a group isomorphic to $\mathbb{Z}_{p^2}$.

Let $J := \sum_{i,j} c(i,j)1_i \otimes 1_j$, $c(i,j) := q^{-i(j-j')}$, where j' denotes the remainder of division of j by p, be the twist in $\mathbb{C}[\mathbb{Z}_{p^2}]^{\otimes 2}$ defined in [G]. Define $\bar{H}$ to be the twist of $\hat{H}$ by J^{-1}: $\bar{H} := \hat{H}^{J^{-1}}$. Since by [G], $dJ = \Phi_s$, $\bar{H}$ is a finite dimensional radically graded **Hopf** algebra. Since the rank of $\bar{H}[1]$ over $\bar{H}[0]$ is > 1, we have at least 2 independent over $\bar{H}[0]$ skew primitive elements $x_1, x_2 \in \bar{H}[1]$ which are eigenvectors for $\mathrm{Ad}(g)$:

$$gx_1g^{-1} = q^{d_1}x_1,\ \Delta(x_1) = x_1 \otimes g^{b_1} + 1 \otimes x_1$$

and

$$gx_2g^{-1} = q^{d_2}x_1,\ \Delta(x_2) = x_2 \otimes g^{b_2} + 1 \otimes x_2.$$

Since $H_0[1] = 0$, d_1, d_2 must be relatively prime to p. Also, since H has minimal dimension, the algebra $\bar{H}$ is generated by g, x_1, x_2.

By [G], the function $\frac{c(i,j)}{c(i-1,j)}q^j$ is $p-$periodic in each variable. Moreover, the coproduct of $\hat{H}$ maps x_i into $\hat{H}\otimes\hat{H}$; thus, similarly to [G], the function $\frac{c(i,j)}{c(i-d_k,j)}q^{b_kj}$ is $p-$periodic in each variable for $k = 1, 2$. Hence the function $\frac{c(i,j)}{c(i-1,j)}q^{(b_k/d_k)j}$ is $p-$periodic in each variable for $k = 1, 2$ (here b_k/d_k is the ratio taken in $\mathbb{Z}_{p^2}$). We thus conclude that $b_k = d_k$ modulo p, for $k = 1, 2$.

Now set $\bar{g} := g^{b_1}$, $\bar{q} := q^{d_1b_1}$, $b := b_2/b_1$ and $d := d_2/d_1$. We obtain

$$\bar{g}x_1\bar{g}^{-1} = \bar{q}x_1,\ \Delta(x_1) = x_1 \otimes \bar{g} + 1 \otimes x_1$$

and

$$\bar{g}x_2\bar{g}^{-1} = \bar{q}^dx_2,\ \Delta(x_2) = x_2 \otimes \bar{g}^b + 1 \otimes x_2,$$

where $b, d \in \mathbb{Z}_{p^2}$ and $b = d$ modulo p.

Extend $\bar{H}$ to a Hopf algebra H' generated by $\bar{H}$ and two commuting grouplike elements g_1, g_2, with relations $g_ix_jg_i^{-1} = \bar{q}^{\delta_{ij}}x_j$, $g_i^{p^2} = 1$ for $i, j = 1, 2$, and $\bar{g} = g_1g_2^d$. (The proof that this is possible is the same as the proof given above of the fact that H can be extended by adjoining g.)

Let $\lambda \in \mathbb{Z}_{p^2}$. Let

$$T = T_\lambda := \sum_{\gamma,\beta} \bar{q}^{\lambda\beta_1\gamma_2}1_\beta \otimes 1_\gamma \in \mathbb{C}[\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}]^{\otimes 2},$$

where $\beta = (\beta_1, \beta_2)$, $\gamma = (\gamma_1, \gamma_2)$ and $\{1_\beta | \beta \in \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}\}$ is the set of primitive idempotents of $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$. This is a Hopf twist. Consider the new coproduct Δ_T, obtained by twisting Δ by T. That is, $\Delta_T(z) = T\Delta(z)T^{-1}$.

Using the facts that $1_\beta g_i = \bar{q}^{\beta_i}1_\beta$ and $1_\beta x_i = x_i1_{\beta-\epsilon_i}$, $i = 1, 2$, where $\epsilon_1 := (1, 0)$ and $\epsilon_2 := (0, 1)$, it is straightforward to verify that

$$\Delta_T(x_1) = x_1 \otimes g_1g_2^{\lambda+d} + 1 \otimes x_1 \text{ and } \Delta_T(x_2) = x_2 \otimes g_1^bg_2^{bd} + g_1^\lambda \otimes x_2.$$

Therefore, if we set

$$z_1 := x_1,\ z_2 := g_1^{-\lambda}x_2,\ h_1 := g_1g_2^{\lambda+d} \text{ and } h_2 := g_1^{b-\lambda}g_2^{bd}$$

we get

$$\Delta_T(z_1) = z_1 \otimes h_1 + 1 \otimes z_1 \text{ and } \Delta_T(z_2) = z_2 \otimes h_2 + 1 \otimes z_2.$$

Now, the relations

$$h_1z_1h_1^{-1}=\bar{q}z_1,\ h_1z_2h_1^{-1}=\bar{q}^{\lambda+d}z_2,\ h_2z_1h_2^{-1}=\bar{q}^{b-\lambda}z_1 \text{ and } h_2z_2h_2^{-1}=\bar{q}^{bd}z_2,$$

imply that the braiding matrix B of $(H')^T$ (in the sense of [AS]) is given by $b_{11}=\bar{q}$, $b_{12}=\bar{q}^{\lambda+d}$, $b_{21}=\bar{q}^{b-\lambda}$ and $b_{22}=\bar{q}^{bd}$.

Now set $\lambda=(b-d)/2$. In this case $b_{12}=b_{21}=\bar{q}^{(b+d)/2}$, so the braiding matrix is symmetric, and the corresponding Nichols algebra is of FL type in the sense of [AS].

According to [AS], the Cartan matrix A corresponding to B has $a_{12}=b+d$ and $a_{21}=(b+d)/bd$ (modulo p^2). Since in our situation $b=d$ modulo p, we get that modulo p, $a_{12}=2b$ and $a_{21}=2/b$, and hence that $a_{12}a_{21}=4$ modulo p. We claim that this implies that the Cartan matrix A cannot be of finite type.

Indeed, in the finite type case ($A_1\times A_1$, A_2, B_2 and G_2), $a_{12}a_{21}=0,1,2,3$. Therefore if $p>3$, A cannot be of finite type. For $p=3$, we get that $a_{12}=a_{21}=-1$ (A_2 case) and $b=1$ modulo 3. But this implies that $b^2+b+1=0$ modulo 9, which leads to a contradiction.

Now by Theorem 1.1 (ii) in [AS], the algebra $(H')^T$ (and hence H') is infinite dimensional. This gives a contradiction and completes the proof of Theorem 3.1.

4. Construction of finite dimensional basic quasi-Hopf algebras

In this section we generalize the construction of $A(q)$ from [G], and construct finite dimensional basic quasi-Hopf algebras which are not twist equivalent to a Hopf algebra.

Let $n\geq 2$ be an integer, and q a primitive root of 1 of order n^2. Let H be a finite dimensional Hopf algebra generated by grouplike elements g_i and skew-primitive elements e_i, $i=1,\dots,m$, such that

$$g_i^{n^2}=1,\ g_ig_j=g_jg_i,\ g_ie_jg_i^{-1}=q^{\delta_{i,j}}e_j$$

and

$$\Delta(e_i)=e_i\otimes K_i+1\otimes e_i,$$

where $K_i:=\prod_j g_j^{a_{ij}}$ for some a_{ij} in $\mathbb{Z}_{n^2}$.

Assume that H has a projection onto $\mathbb{C}[(\mathbb{Z}_{n^2})^m]$, $g_i\mapsto g_i$ and $e_i\mapsto 0$, and let $B\subset H$ be the subalgebra generated by $\{e_i\}$. Then by Radford's theorem [R], the multiplication map $\mathbb{C}[(\mathbb{Z}_{n^2})^m]\otimes B\to H$ is an isomorphism of vector spaces. Therefore, $A:=\mathbb{C}[(\mathbb{Z}_n)^m]B\subset H$ is a subalgebra of dimension $\dim(H)/n^m$. It is generated by g_i^n and e_i.

Let $\{1_\beta|\beta=(\beta_1,\dots,\beta_m)\in(\mathbb{Z}_{n^2})^m\}$ be the set of primitive idempotents of $\mathbb{C}[\mathbb{Z}_{n^2}^m$, and denote by $\epsilon_i\in(\mathbb{Z}_{n^2})^n$ the vector with 1 in the ith place and 0 elsewhere. Note that

$$1_\beta g_i=q^{\beta_i}1_\beta \text{ and } 1_\beta e_i=e_i1_{\beta-\epsilon_i}.$$

Let $c(z,y)$ be the coefficients of the twist J as above introduced in [G]. Recall from [G] that $c(z,y)=q^{-z(y-y')}$, where y' denotes the remainder of division of y by n.

Let

$$\mathbb{J}:=\sum_{\beta,\gamma\in(\mathbb{Z}_{n^2})^m}\prod_{i,j=1}^{m}c(\beta_i,\gamma_j)^{a_{ij}}1_\beta\otimes 1_\gamma.$$

It is clear that it is invertible and $(\varepsilon \otimes \mathrm{id})(\mathbb{J}) = (\mathrm{id} \otimes \varepsilon)(\mathbb{J}) = 1$. Define a new coproduct $\Delta_{\mathbb{J}}(z) = \mathbb{J}\Delta(z)\mathbb{J}^{-1}$.

Lemma 4.1. *The elements $\Delta_{\mathbb{J}}(e_i)$ belong to $A \otimes A$.*

Proof. This lemma for $m = 1$ was proved in [G]. The general case follows from the case $m = 1$ by a straightforward computation. □

Lemma 4.2. *The associator $\Phi := d\mathbb{J}$ obtained by twisting the trivial associator by $\mathbb{J}$ is given by the formula*

$$\Phi = \sum_{\beta,\gamma,\delta \in \mathbb{Z}_n^m} \left(\prod_{i,j=1}^{m} q^{a_{ij}\beta_i((\gamma_j+\delta_j)'-\gamma_j-\delta_j)} \right) \mathbf{1}_\beta \otimes \mathbf{1}_\gamma \otimes \mathbf{1}_\delta,$$

where $\{\mathbf{1}_\beta\}$ are the primitive idempotents of $\mathbb{Z}_n^m$ ($\mathbf{1}_\beta g_i^n = q^{n\beta_i}\mathbf{1}_\beta$), and we regard the components of β, γ, δ as elements of $\mathbb{Z}$. Thus Φ belongs to $A \otimes A \otimes A$.

Proof. One has

$$\Phi = \sum_{\beta,\gamma,\delta \in (\mathbb{Z}_{n^2})^m} \prod_{i,j=1}^{m} \left(\frac{c(\beta_i+\gamma_i,\delta_j)c(\beta_i,\gamma_j)}{c(\beta_i,\gamma_j+\delta_j)c(\gamma_i,\delta_j)} \right)^{a_{ij}} 1_\beta \otimes 1_\gamma \otimes 1_\delta.$$

Substituting the expression of $c(z, y)$, similarly to [G] we get the statement. □

Thus we get our second main result.

Theorem 4.3. *The algebra A is a quasi-Hopf subalgebra of $H^{\mathbb{J}}$, which has coproduct $\Delta_{\mathbb{J}}$ and associator Φ.*

Proof. We have shown that $\Delta_{\mathbb{J}} : A \to A \otimes A$ and $\Phi \in A \otimes A \otimes A$. It is also straightforward to show that $S_{\mathbb{J}} : A \to A$ and $\alpha \in A$ if β is gauged to be 1 (where $S_{\mathbb{J}}$, α, and β are the antipode and the distinguished elements of $H^{\mathbb{J}}$). Thus A is a quasi-Hopf subalgebra of $H^{\mathbb{J}}$. □

This yields many examples of finite dimensional basic quasi-Hopf algebras A. For instance, let $\mathfrak{g}$ be a finite dimensional simple Lie algebra, and $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$. Assume that n is odd, coprime to 3 if $\mathfrak{g} = G_2$. Then we can take H to be the Frobenius-Lusztig kernel $u_q(\mathfrak{b})$. In this case, A is a quasi-Hopf algebra of dimension $n^{\dim \mathfrak{g}}$. Another example is obtained from $H = \mathrm{gr}(u_q(\mathfrak{g}))$ (with respect to the coradical filtration).

Remark 4.4. If for some i, $a_{ii} \neq 0$ modulo n, then A is not twist equivalent to a Hopf algebra. Indeed, the associator Φ is non-trivial since the $3-$cocycle corresponding to Φ restricts to a non-trivial $3-$cocycle on the cyclic group $\mathbb{Z}_n$ consisting of all tuples whose coordinates equal 0, except for the ith coordinate. Since A projects onto $(\mathbb{C}[\mathbb{Z}_n^m], \Phi)$ with non-trivial Φ, A is not twist equivalent to a Hopf algebra.

For instance, this is the case in the above two examples obtained from $u_q(\mathfrak{b})$ and $u_q(\mathfrak{g})$.

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